On commutation relations in quantum mechanics

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The primary role of a commutation relation between two operators is that it is a measure of the compatibility of the operators when they are measured sequentially. That is, the measurement of an operator disturbs the prior measurement of another operator unless the operators commute. Commutation relation also serve as a statement of the derivative of an operator with respect to a parameter. Thus, commutation relations encode the dynamics of a system. The third role of a commutation relation is that they provide the algebra for the composition of generators. These widely known ideas are emphasized and illustrated here with examples.

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I. INTRODUCTION

In Newtonian mechanics we associate dynamics with an evolution in time, a movement in space, or a change in orientation. An underlying assumption is that we can model physical quantities as a mathematical function. The dynamics is then a variation of the function with respect to a variable, say λ, using the idea of derivative in infinitesimal calculus. This basic principle that has had enormous success in understanding planetary motion, falls short when it comes to modeling an intrinsic property of an elementary particle, like mass or spin, because intrinsic properties by definition cannot be represented by a function. This was one of many reasons for the advent of quantum mechanics. For example, the spin S of an electron is expressed in terms of the reduced Planck constant $\hbar$ as

$$S = \frac{\hbar}{2}, \tag{1}$$

which is a universal constant and thus does not change as time evolves, or because the electron moves in space, or when the electron changes its orientation. However, the spin S of an electron is the magnitude of a vector and the components of this vector are expected to depend on the orientation of the laboratory around it. The projection of this vector along the vertical z direction of the laboratory, [that has definite values in a measurement after we have measured the magnitude of the spin,] is given by a 2×2 matrix

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

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Newtonian mechanics:	λ	$f(\lambda)$	$\frac{df}{d\lambda} = \lim_{\Delta\lambda\to 0}\frac{\Delta f}{\Delta\lambda}$
Hamiltonian mechanics:	G	X	$\frac{dX}{d\lambda} = \big[X,G\big]_{\text{P.B.}}$
Quantum mechanics:	G	X	$\frac{dX}{d\lambda} = \frac{1}{i\hbar}\big[X,G\big]$

TABLE I. Derivative of a function in infinitesimal calculus versus derivative of a dynamical variable in Hamiltonian mechanics and derivative of an operator in quantum mechanics. Here G is the generator of a transformation that captures the variation of an operator X with respect to variable λ. Here $\big[X,G\big]_{\text{P.B.}}$ is the Poisson bracket and $\big[X,G\big]$ is the commutator.

Observe markedly that the quantities in Eqs. (1) and (2) have no explicit dependence on angles and thus are not mathematical functions in the traditional sense. The angular dependence in the spin of an electron can be understood as that of an implicit variable.

II. DYNAMICS: IMPLICIT CALCULUS

Physical quantities with no explicit dependence on variables like time t, position $\mathbf{r}$, and orientation $\boldsymbol{\theta}$, are called dynamical variables in Hamiltonian mechanics and represented by operators in quantum mechanics. The derivative of a dynamical variable with respect to an implicit variable is given using the mathematical framework of Lie algebra. In Hamiltonian mechanics these derivatives are defined in terms of Poisson brackets and in quantum mechanics in terms of commutation relations, both of which satisfy the axioms of a Lie algebra. This is summarized in Table I. It is fitting to mention in this context that the operation of a vector product between two vectors also satisfy the axioms of Lie algebra. In a Lie algebra the change in a quantity X associated with evolution in time t is described by the Hamiltonian operator H that generates the changes in X as a transformation that is a translation of the origin of time. Thus,

the Hamiltonian is called the generator of evolution in time. Similarly, movement in space $\mathbf{r}$ is generated by the linear momentum operator $\mathbf{P}$ as a transformation that translates the spatial origin. Angular momentum operator $\mathbf{J}$ is the generator of change in orientation $\boldsymbol{\theta}$ of an object through the transformation associated to rotations in space. Since the advent of special relativity we are also interested in changes with respect to the relative velocity $\mathbf{v}$ of reference frames which are generated by the boost operator $\mathbf{N}$. The variables we have listed, in terms of their respective transformations, constitute the Poincaré transformation, see Table II. This is only a limited set of variables that are used to characterize dynamics.

The role of commutation relations in quantum mechanics as a derivative operation has its foundational basis [1] in the role of Poisson brackets serving as derivatives in Hamiltonian mechanics. In this article we emphasize this role,

$$\frac{dX}{d\lambda} = \frac{1}{i\hbar}\big[X, G\big], \tag{3}$$

as the infinitesimal variation in an operator X due to an infinitesimal transformation $\delta\lambda$ generated by the generator G associated to λ [2, 3], where $\hbar = h/(2\pi)$ is constructed from the Planck constant h that has units of angular momentum. The quantities X and G are abstract operator constructions in a Hilbert space, while the parameter λ describing the transformation takes on numerical values on a real line. The commutation relation in Eq. (3) as a statement of the derivative should be contrasted with the traditional definition of a derivative in infinitesimal calculus. In particular, commutation relation

$$\big[X, G\big] = XG - GX, \tag{4}$$

seems to be characterized by ordering of operations and does not seem to immediately suggest a connection with the continuous limiting process of infinitesimal calculus. The mathematical formalism of quantum mechanics in this spirit was developed by Heisenberg, Jordan, and Born, and was already used by Pauli [4, 5] in year 1926 to determine the hydrogen spectrum. This development was sidelined by the development of Schrödinger's equation and wave mechanics.

A. Examples

We illustrate the above ideas in the case of three examples.

1. *Uniform acceleration*

As our first illustrative example let us consider the motion of an object of mass m when it is dropped in a region of uniform gravity g in the vertically downward direction. We do not associate quantum effects to a basketball dropped on the surface of earth, however we expect quantum effects to be relevant when the mass of the ball is significantly smaller. Assuming no air resistance and the vertical direction to be along z axis the Hamiltonian for the system is

$\delta\lambda$	Transformation	Generator	G
δt	Time evolution	Hamiltonian	H
$\delta\mathbf{r}$	Spatial translation	Linear momentum	$\mathbf{P}$
$\delta\boldsymbol{\theta}$	Spatial rotation	Angular momentum	$\mathbf{J}$
$\delta\mathbf{v}$	Special relativity	Boost	$\mathbf{N}$

TABLE II. Transformation of a coordinate system into another due to, a translation of the spatial origin, a translation of the time origin, a rotation of the space axes, a constant relative velocity between the two systems, and the associated generators which together comprise the 10-parameter Poincaré transformation.

$$H = \frac{p^2}{2m} + mgz, \tag{5}$$

where z is vertical position of the mass and p is the momentum of mass. If position z and momentum p are operators, using Eq. (3) we can write

$$\frac{dz}{dz} = \frac{1}{i\hbar}\big[z, p\big] = 1, \tag{6}$$

which is Heisenberg's canonical commutation relation. The dynamical evolution of the position of the object is completely determined in terms of time derivatives at $t = 0$ using Taylor expansion,

$$z(t) = z(0) + \frac{t}{1!}\left\{\frac{dz}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2z}{dt^2}\right\}_{t=0} + \ldots, \tag{7}$$

where all the derivatives are given using Eq. (3). The first derivative of position, the velocity, is

$$\begin{aligned}
\frac{dz}{dt} &= \frac{1}{i\hbar}\big[z, H\big] && \text{(8a)}\\
&= \frac{1}{i\hbar}\big[z, \frac{p^2}{2m} + mgz\big] && \text{(8b)}\\
&= \frac{1}{2m}\frac{1}{i\hbar}\big[z, p^2\big] && \text{(8c)}\\
&= \frac{p}{m}, && \text{(8d)}
\end{aligned}$$

where in the last step we used Eq. (6) after employing the distribution property of commutation relation. The relation we derived for velocity in terms of momentum and mass is true at the classical level and quantum level. Note that the derivation is not using the traditional idea

of derivative and is instead using the commutation relations alone. The second derivative of position, the acceleration, is

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\frac{dz}{dt} = \frac{d}{dt}\frac{p}{m} \tag{9a}$$

$$= \frac{1}{m}\frac{1}{i\hbar}\big[p, H\big] \tag{9b}$$

$$= \frac{1}{m}\frac{1}{i\hbar}\big[p, mgz\big] \tag{9c}$$

$$= -g, \tag{9d}$$

where in the first step we used the equation for velocity and in the last step we used Eq. (6). Again, the equation for acceleration is the same as in classical mechanics. All the higher derivatives evaluate to zero if we assume that the acceleration of due to gravity has no implicit time dependence, that is,

$$\frac{d^nz}{dt^n} = 0, \qquad n \geq 3. \tag{10}$$

Using these derivatives, for $t = 0$, in the Taylor expansion in Eq. (7), we obtain the time evolution of the position operator given by

$$z(t) = z(0) + p(0)\frac{t}{m} - \frac{1}{2}gt^2. \tag{11}$$

Similarly, we can derive

$$p(t) = p(0) - mg\,t. \tag{12}$$

Remarkably, the classical equations of motion hold even quantum mechanically. However, all is not what it seems. The difference lies in the specification of the initial conditions $z(0)$ and $p(0)$. Unlike in Newtonian mechanics, we cannot specify the initial conditions to unlimited certainty in quantum mechanics. That is, there is a spread in the specifications of $z(0)$ and $p(0)$. Let us prepare the system such that it is in the minimum uncertainty state

$$\frac{\hbar}{2} = \delta z(0)\delta p(0) \tag{13}$$

at $t = 0$. Using the above equations we can evaluate the commutation relation

$$\big[z(0), z(t)\big] = \big[z(0), p(0)\big]\frac{t}{m} = i\hbar\frac{t}{m}. \tag{14}$$

Thus, the spread in the position and momentum of these objects, $\delta z(t)$ and $\delta p(t)$, in a beam of identical objects, using generalized Heisenberg uncertainty relation in Eq. (14) is

$$\frac{\hbar t}{2m} \leq \delta z(0)\delta z(t). \tag{15}$$

This spread can be further quantified by evaluating the transformation function $\langle z', 0|z', t\rangle$ that is interpreted as a probability amplitude or the wave function [3] which we shall not pursue here.

2. *Harmonic oscillator*

As our second illustrative example we consider a harmonic oscillator described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2, \tag{16}$$

where x is the extension in the spring of spring constant k and p is the momentum of the mass m attached to the spring. Heisenberg's canonical commutation relation is obtained as

$$\frac{dx}{dx} = \frac{1}{i\hbar}\big[x, p\big] = 1. \tag{17}$$

The velocity is given by

$$\frac{dx}{dt} = \frac{1}{i\hbar}\big[x, H\big] = \frac{p}{m}. \tag{18}$$

The acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\frac{dx}{dt} = \frac{d}{dt}\frac{p}{m} = \frac{1}{m}\frac{1}{i\hbar}\big[p, H\big] = -\omega^2 x, \tag{19}$$

where $\omega = \sqrt{k/m}$ is the characteristic frequency of the spring-mass system. Similarly, we can evaluate the third and fourth derivatives of position as

$$\frac{d^3x}{dt^3} = -\frac{\omega^2}{m}p, \tag{20}$$

$$\frac{d^4x}{dt^4} = \omega^4 x, \tag{21}$$

and we can evaluate higher derivatives iteratively. Then, using Taylor expansion we have

$$x(t) = x(0) + \frac{t}{1!}\left\{\frac{dx}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2x}{dt^2}\right\}_{t=0} + \ldots, \tag{22}$$

which is equal to

$$x(t) = x(0)\left[1 - \frac{(\omega t)^2}{2!} + \ldots\right] + \frac{p(0)}{m\omega}\left[\frac{(\omega t)}{1!} - \frac{(\omega t)^3}{3!} + \ldots\right] \tag{23}$$

$$= x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t. \tag{24}$$

Again, the equations of motion are identical to those in classical physics. However, their interpretation have quantum features because of the uncertainty in the quantities that do not commute. For example,

$$\big[x(0), x(t)\big] = \big[x(0), p(0)\big]\frac{\sin\omega t}{m\omega} = i\hbar\frac{\sin\omega t}{m\omega}. \tag{25}$$

The implication is that

$$\frac{\hbar}{2}\frac{\sin\omega t}{m\omega} \leq \delta x(0)\delta x(t). \tag{26}$$

We can reproduce the case of small time t using the limit $\omega t \to 0$ to yield

$$\frac{\hbar t}{2m} \leq \delta x(0)\delta x(t). \tag{27}$$

3. Rotation of spin

Our third example, starting with the paragraph containing Eq. (96) in Section IV A, illustrates how a Hamiltonian describing the interaction of a spin in a magnetic field transforms under a transformation generated by a rotation.

III. UNCERTAINTY

Another role of the commutation relation between two operators is that it characterizes the incompatibility of the corresponding observables when they are measured sequentially. Unless two operators commute, a measurement of an observable corresponding to operator A generally disturbs the outcome of a subsequent measurement of the observable corresponding to operator B. This is called an uncertainty relation. We present here the derivation of uncertainty relation as per Ref. [3].

Let A and B be Hermitian operators. The expectation (or average) values of A and B in the physical state $|\ \rangle$ will be denoted by

$$\langle A\rangle = \langle\ |A|\ \rangle, \qquad \langle B\rangle = \langle\ |B|\ \rangle, \tag{28}$$

and the mean square deviation from these averages as

$$(\delta A)^2 = \langle\ |(A-\langle A\rangle)^2|\ \rangle \equiv \langle 1|1\rangle, \tag{29a}$$
$$(\delta B)^2 = \langle\ |(B-\langle B\rangle)^2|\ \rangle \equiv \langle 2|2\rangle, \tag{29b}$$

where

$$\langle 1| = \langle\ |(A-\langle A\rangle), \qquad |2\rangle = (B-\langle B\rangle)|\ \rangle. \tag{30}$$

Using the unitary counterpart of the 3-dimensional Euclidean statement $(\mathbf{a}\cdot\mathbf{b})^2 \le (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})$, or $\cos^2\theta \le 1$, we have

$$|\langle 1|2\rangle|^2 \le \langle 1|1\rangle\langle 2|2\rangle, \tag{31}$$

where the equality applies only when $|1\rangle$ is 'parallel' to $|2\rangle$. Thus, the mean square deviations in Eqs. (29) satisfy

$$|\langle 1|2\rangle| \le (\delta A)(\delta B). \tag{32}$$

The displaced operators,

$$X = A - \langle A\rangle, \qquad Y = B - \langle B\rangle, \tag{33}$$

are also Hermitian. Though the bilinear constructions, XY and YX, are not Hermitian, the following symmetric and anti-symmetric products built out of these bilinear constructions

$$D = XY + YX, \qquad D^\dagger = D, \tag{34a}$$
$$iC = XY - YX, \qquad C^\dagger = C, \tag{34b}$$

are Hermitian. The anti-symmetric product,

$$XY - YX = \big[X, Y\big] \tag{35}$$

which is in general not Hermitian, is identified as the commutation relation between two operators. Note the role of i in Eq. (34b) in rendering the anti-symmetric construction Hermitian. We have,

$$C = \frac{1}{i}\big[X, Y\big] = \frac{1}{i}\big[A, B\big]. \tag{36}$$

The non-Hermitian operators can thus be expressed in terms of the above Hermitian operators as

$$2XY = D + iC, \qquad (XY)^\dagger = YX, \tag{37a}$$
$$2YX = D - iC, \qquad (YX)^\dagger = XY. \tag{37b}$$

The projection of states, $\langle 1|2\rangle$, in Eqs. (31) and (32), can be expressed as an expectation value of a bilinear construction of the displaced operators as

$$\langle 1|2\rangle = \langle\ |XY|\ \rangle \tag{38}$$

and thus can be expressed in terms of Hermitian operators as

$$2\langle 1|2\rangle = \langle D\rangle + i\langle C\rangle. \tag{39}$$

Since expectation values of Hermitian operators are real we have

$$\langle D\rangle^* = \langle D\rangle \qquad \langle C\rangle^* = \langle C\rangle. \tag{40}$$

We have thus given a complex number representation of the projection of states, $\langle 1|2\rangle$, where the real part and imaginary parts are given in terms of expectation values of bilinear constructions of the displaced operators. Then, using the Pythagorean theorem on the complex plane we deduce

$$\langle D\rangle^2 + \langle C\rangle^2 = 4|\langle 1|2\rangle|^2. \tag{41}$$

This, then, implies the inequality

$$\frac{|\langle C\rangle|}{2} \le |\langle 1|2\rangle|. \tag{42}$$

In the inequalities we presented in Eqs. (42) and (32) we have found lower and upper bounds on $|\langle 1|2\rangle|$ which together reads

$$\frac{|\langle C\rangle|}{2} \le |\langle 1|2\rangle| \le (\delta A)(\delta B). \tag{43}$$

Dropping reference to the projection of states and writing the inequality in terms of the lower and upper bounds leads us to the Robertson's generalization to Heisenberg's uncertainty relation [6]

$$\frac{|\langle C\rangle|}{2} \le (\delta A)(\delta B). \tag{44}$$

That is, the expectation value of the commutation relation between two operators is a lower bound for the product of uncertainties in the two operators. Applying

this to the pair of operators (x, p) we identify the Heisenberg's uncertainty relation [7]

$$\frac{\hbar}{2} \leq (\delta x)(\delta p) \tag{45}$$

after using

$$\frac{1}{i}\big[x, p\big] = \hbar. \tag{46}$$

An uncertainty relation is called a minimal uncertainty when the equality in the above relations is satisfied.

Uncertainty relations involving three Pauli matrices is presented in the following discussion.

A. Uncertainty relations for spin

The Pauli matrices σ_i are traceless Hermitian matrices that satisfy the algebra

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k, \tag{47}$$

where i, j, are either $1, 2$, or 3. The Kronecker δ-function δ_{ij} evaluates 1 when $i = j$, and 0 otherwise. The Levi-Cevita symbol ε_{ijk} is defined to be 1 for ε_{123} and for all the even permutation obtained then on, -1 for ε_{213} and for all the even permutation obtained then on, and 0 otherwise. The algebra in Eq. (47) can be used to derive the commutation relations

$$\big[\sigma_i, \sigma_j\big] = 2i\varepsilon_{ijk}\sigma_k \tag{48}$$

and the anti-commutation relations

$$\big\{\sigma_i, \sigma_j\big\} = 2\delta_{ij}. \tag{49}$$

If the components α_i of a vector operator $\boldsymbol{\alpha}$ do not commute, then $\boldsymbol{\alpha} \times \boldsymbol{\alpha} \neq 0$ in general. In this spirit, the commutation relations for Pauli matrices satisfy

$$\frac{\boldsymbol{\sigma}}{2} \times \frac{\boldsymbol{\sigma}}{2} = i\frac{\boldsymbol{\sigma}}{2}. \tag{50}$$

This is a special case of the angular momentum statement

$$\mathbf{J} \times \mathbf{J} = i\hbar\mathbf{J}. \tag{51}$$

The Pauli matrices of Eq. (47), in the eigenbasis of σ_3, have the representation

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{52}$$

Consider a normalized state

$$|\,\rangle = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 \\ y_1 + iy_2 \end{pmatrix}, \qquad |u|^2 + |v|^2 = 1. \tag{53}$$

Since the Pauli matrices are Hermitian, the expectation value of Pauli matrices are necessarily real. That is,

$$\langle\boldsymbol{\sigma}\rangle = \langle\,|\boldsymbol{\sigma}|\,\rangle = \langle\boldsymbol{\sigma}\rangle^*. \tag{54}$$

The expectation values of the individual Pauli matrices with respect to the normalized state in Eq. (53),

$$\langle\sigma_x\rangle = \langle\,|\sigma_x|\,\rangle = u^*v + v^*u, \tag{55a}$$
$$\langle\sigma_y\rangle = \langle\,|\sigma_y|\,\rangle = -i(u^*v - v^*u), \tag{55b}$$
$$\langle\sigma_z\rangle = \langle\,|\sigma_z|\,\rangle = |u|^2 - |v|^2, \tag{55c}$$

is not very insightful. However, when we calculate the squares of the expectation values, recall these are real numbers,

$$\langle\sigma_x\rangle^2 = u^{*2}v^2 + u^2v^{*2} + 2|u|^2|v|^2, \tag{56a}$$
$$\langle\sigma_y\rangle^2 = -u^{*2}v^2 - u^2v^{*2} + 2|u|^2|v|^2, \tag{56b}$$
$$\langle\sigma_z\rangle^2 = |u|^4 + |v|^4 - 2|u|^2|v|^2, \tag{56c}$$

it is intriguing when we note that the sum of the squares of the expectation value of the Pauli matrices is always unity. That is, if the three expectation values were interpreted as three components of a vector, then this vector is constrained to remain on a sphere of unit radius,

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1. \tag{57}$$

We can thus conclude that the magnitude of the expectation value of each of the Pauli matrix is less than or equal to unity. These expectation values are constrained to remain on a sphere, and an implication is that the three expectation values can not be simultaneously zero.

In physical models observables are described in terms of Hermitian operators like the Pauli matrices, and the expectation values of these operators are associated to a measurement. The error in a measurement is defined using the idea of variance in statistics. The squared-error in the measurement of the Pauli matrices are defined as

$$(\delta\sigma_x')^2 = \langle\,|\big\{\sigma_x - \langle\sigma_x\rangle\big\}^2|\,\rangle, \tag{58a}$$
$$(\delta\sigma_y')^2 = \langle\,|\big\{\sigma_y - \langle\sigma_y\rangle\big\}^2|\,\rangle, \tag{58b}$$
$$(\delta\sigma_z')^2 = \langle\,|\big\{\sigma_z - \langle\sigma_z\rangle\big\}^2|\,\rangle. \tag{58c}$$

Following Dirac [1] and Schwinger [3], we used primes to denote eigenvalues of the respective operators, or to represent non-operator quantities. Using the properties of Pauli matrices we can immediately show that

$$(\delta\sigma_x')^2 = 1 - \langle\sigma_x\rangle^2, \tag{59a}$$
$$(\delta\sigma_y')^2 = 1 - \langle\sigma_y\rangle^2, \tag{59b}$$
$$(\delta\sigma_z')^2 = 1 - \langle\sigma_z\rangle^2. \tag{59c}$$

Thus, we conclude that the errors in measurement of each of the Pauli matrix is less than or equal to unity. Adding the squared-errors of Eqs. (59) and using the constraint in Eq. (57) we can show that

$$(\delta\sigma_x')^2 + (\delta\sigma_y')^2 + (\delta\sigma_z')^2 = 2. \tag{60}$$

Thus, the errors are constrained on a sphere of radius 2. This time the implication is that the three errors can not be simultaneously zero. Thus, Eq. (60) is a statement of

unavoidable uncertainty in measurement of Pauli matrices.

Heisenberg's uncertainty relation [7] concerns position and momentum. Using Robertson's [6] generalization of Heisenberg's uncertainty relation,

$$\frac{1}{2}|\langle C\rangle| \leq (\delta A')(\delta B'), \tag{61}$$

for Hermitian operators A and B, such that $[A, B] = iC$, we can deduce the uncertainty relations for the Pauli matrices to be

$$\langle\sigma_x\rangle^2 \leq (\delta\sigma_y')^2(\delta\sigma_z')^2, \tag{62a}$$
$$\langle\sigma_y\rangle^2 \leq (\delta\sigma_z')^2(\delta\sigma_x')^2, \tag{62b}$$
$$\langle\sigma_z\rangle^2 \leq (\delta\sigma_x')^2(\delta\sigma_y')^2. \tag{62c}$$

These uncertainty relations when added together gives an uncertainty relation involving all the three Pauli matrices,

$$1 \leq (\delta\sigma_x')^2(\delta\sigma_y')^2 + (\delta\sigma_y')^2(\delta\sigma_z')^2 + (\delta\sigma_z')^2(\delta\sigma_x')^2. \tag{63}$$

B. Minimal uncertainty

The minimal uncertainty in each of the above relations in Eqs. (62) and Eq. (63) is attained if

$$\langle\sigma_x\rangle\langle\sigma_y\rangle = 0, \tag{64a}$$
$$\langle\sigma_y\rangle\langle\sigma_z\rangle = 0, \tag{64b}$$
$$\langle\sigma_z\rangle\langle\sigma_x\rangle = 0. \tag{64c}$$

In conjunction with the constraint imposed on the expectation values in Eq. (57), we learn that all three of the expectation values of Pauli matrices can not be zero for a given state. However, two of them can be zero, which is sufficient for minimizing all the uncertainty relations in Eqs. (62) and Eq. (63). For example, a state satisfying

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = 0 \tag{65}$$

minimizes all the four uncertainty relations. The state

$$|\text{min}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{66}$$

is such a state. When the system is in this state, we have

$$\langle\sigma_x\rangle = 0, \qquad \delta\sigma_x' = 1, \tag{67a}$$
$$\langle\sigma_y\rangle = 0, \qquad \delta\sigma_y' = 1, \tag{67b}$$
$$\langle\sigma_z\rangle = 1, \qquad \delta\sigma_z' = 0. \tag{67c}$$

IV. SPIN

It is energetically most favorable for a magnet to orient its magnetic dipole moment $\boldsymbol{\mu}$ in the direction of a magnetic field $\mathbf{B}$. This interaction energy is governed by the Hamiltonian

$$H = -\boldsymbol{\mu}\cdot\mathbf{B}. \tag{68}$$

The magnetic moment of a particle of charge q and mass m is given in terms of the angular momentum $\mathbf{J}$ of the particle,

$$\boldsymbol{\mu} = g\,\frac{q}{2m}\,\mathbf{J}, \tag{69}$$

where g, the g-factor, is a measure of the composite nature of a particle. For example[1], g-factor is exactly 2 for an electron suggesting that it has no structure within our experimental scope, g is 5.6 for a proton which is known to be composed of gluons and three quarks, and $g \sim 2.0$ for a muon which is the most accurately measured parameter by mankind. The product of g and $q/2m$ is a measure of the ratio of magnetic moment to angular momentum and is called the gyromagnetic ratio. The spin of an elementary particle is an intrinsic property of the particle. For spin-1/2 particles it is equal to

$$\mathbf{J} = \frac{\hbar}{2}\boldsymbol{\sigma}, \tag{70}$$

where $\hbar$ is rationalized Planck constant and $\boldsymbol{\sigma}$ is intrinsic in that it is independent of the position of the particle. Thus, the suggestion is that every spin-1/2 particle is a magnet with a Hamiltonian

$$H = -H_0\,(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}), \qquad H_0 = g\,\frac{q}{2m}\,\frac{\hbar}{2}, \tag{71}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector representing the direction of the magnetic field at the position of the particle. In particular, even though the magnetic field and the angular momentum are vectors in our three-dimensional space, the angular momentum has an additional structure in an abstract two-dimensional space. As a consequence, the Hamiltonian is a scalar in our three-dimensional space, with an additional 2×2 matrix structure from the spin. The quantity $(\hat{\mathbf{r}}\cdot\boldsymbol{\sigma})$ is remarkable, because it has no dimensions, involves a direction $\hat{\mathbf{r}}$ in our physical space, and interacts with spin that has a structure $\boldsymbol{\sigma}$ in the internal space that manifests itself as angular momentum $\mathbf{J}$ in our physical space.

For simplicity, we restrict our analysis to spin-1/2 particles. Using index notation for representing the component of vectors in the rectangular basis we have

$$\hat{\mathbf{r}} \to x_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{72a}$$
$$\boldsymbol{\sigma} \to \sigma_i = (\sigma_1, \sigma_2, \sigma_3), \tag{72b}$$

[1] The quantity $(g-2)/2$ for a point-like spin-1/2 particle will be 0. An electron to our current knowledge is such a particle.Values of g has been intentionally listed to only two significant digits here to capture its role as an indicator of compositeness.

where θ and ϕ are points on a unit sphere in our three-dimensional space used to specify the direction of magnetic field and σ_i are Pauli matrices defined in Eq. (52) that is a vector in our three-dimensional space with an additional structure in the abstract two-dimensional space. The Hamiltonian has the form

$$H = -H_0 \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \tag{73}$$

which brings out the explicit 2×2 matrix structure of the energy. The matrix structure in index notation is

$$(\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}) = (x_i \sigma_i) \tag{74}$$

and constitutes of traceless Hermitian 2×2 matrices.

Since we choose the magnetic field in the z-x plane that is perpendicular to the direction of beam along y-axis we have

$$H = -H_0 \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{75}$$

where θ is given by $\hat{\mathbf{z}} \cdot \mathbf{B} = B\cos\theta$. Thus, we have

$$(x_i \sigma_i) = \cos\theta\, \sigma_3 + \sin\theta\, \sigma_1. \tag{76}$$

A. Rotation of spin

Under a rotation about the y-axis the direction of the magnetic field transforms as

$$\hat{\mathbf{r}}(\theta) = \mathbf{R}(\theta) \cdot \hat{\mathbf{r}}(0), \tag{77}$$

or

$$\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{78}$$

or in index notation

$$\bar{x}_i = R_{ij} x_j. \tag{79}$$

How does the Hamiltonian transform under this transformation?

We shall show that a transformation of the form

$$\bar{H} = u H u^\dagger, \tag{80}$$

where

$$u = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{81}$$

in the space constituting of 2×2 traceless Hermitian matrices generates a rotation about the y-axis in our three-dimensional space. Starting from Eq. (80) we have

$$\bar{x}_i \sigma_i = u (x_j \sigma_j) u^\dagger \tag{82}$$

which together with Eq. (79) leads to

$$R_{ij} \sigma_i = u \sigma_j u^\dagger. \tag{83}$$

To isolate the components of rotation matrix R_{ij} we multiply by σ_k and take the trace,

$$R_{ij} \mathrm{tr}(\sigma_i \sigma_k) = \mathrm{tr}(u\, \sigma_j u^\dagger \sigma_k). \tag{84}$$

Using Eq. (47) we then have

$$R_{ij} = \frac{1}{2} \mathrm{tr}(\sigma_i\, u\, \sigma_j u^\dagger) \tag{85}$$

after using the cyclic property of trace. For the particular u in Eq. (81) we can write

$$u = \cos\frac{\theta}{2} + i\sigma_2 \sin\frac{\theta}{2} \tag{86}$$

and

$$u^\dagger = \cos\frac{\theta}{2} - i\sigma_2 \sin\frac{\theta}{2} \tag{87}$$

so that

$$R_{ij} = \frac{1}{2} \mathrm{tr} \left[\sigma_i \left(\cos\frac{\theta}{2} + i\sigma_2 \sin\frac{\theta}{2} \right) \times \sigma_j \left(\cos\frac{\theta}{2} - i\sigma_2 \sin\frac{\theta}{2} \right) \right]. \tag{88}$$

This can be expanded as

$$R_{ij} = \frac{1}{2} \left[\mathrm{tr}(\sigma_i \sigma_j) \cos^2\frac{\theta}{2} - i\,\mathrm{tr}(\sigma_i \sigma_j \sigma_2) \cos\frac{\theta}{2} \sin\frac{\theta}{2} + i\,\mathrm{tr}(\sigma_i \sigma_2 \sigma_j) \cos\frac{\theta}{2} \sin\frac{\theta}{2} + \mathrm{tr}(\sigma_i \sigma_2 \sigma_j \sigma_2) \sin^2\frac{\theta}{2} \right]. \tag{89}$$

Then, using the identities

$$\mathrm{tr}(\sigma_i \sigma_j) = 2\delta_{ij}, \tag{90a}$$

$$\mathrm{tr}(\sigma_i \sigma_j \sigma_k) = 2i\varepsilon_{ijk}, \tag{90b}$$

$$\mathrm{tr}(\sigma_i \sigma_m \sigma_j \sigma_n) = 2\delta_{im}\delta_{jn} - 2\delta_{ij}\delta_{mn} + 2\delta_{in}\delta_{jm}, \tag{90c}$$

we obtain

$$R_{ij} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{91}$$

Thus, we have verified that a rotation about the y-axis in the three-dimensional space, generated by Eq. (91), corresponds to a 'rotation' in the space of traceless Hermitian matrices of the form in Eq. (81). Observe that periodicity in u is attained only when θ ranges from 0 to 4π due to the half-angle dependence in Eq. (81). For

completeness, we list the following correspondence in rotations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \leftarrow \begin{pmatrix} \cos\frac{\phi_1}{2} & i\sin\frac{\phi_1}{2} \\ i\sin\frac{\phi_1}{2} & \cos\frac{\phi_1}{2} \end{pmatrix}, \tag{92a}$$

$$\begin{pmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \leftarrow \begin{pmatrix} \cos\frac{\phi_2}{2} & \sin\frac{\phi_2}{2} \\ -\sin\frac{\phi_2}{2} & \cos\frac{\phi_2}{2} \end{pmatrix}, \tag{92b}$$

$$\begin{pmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} e^{i\frac{\phi_3}{2}} & 0 \\ 0 & e^{-i\frac{\phi_3}{2}} \end{pmatrix}. \tag{92c}$$

The three independent rotations can be expressed in the form, with no summation on repeated indices,

$$u_i = e^{i\sigma_i\frac{\phi_i}{2}} = \cos\frac{\phi_i}{2} + i\sigma_i \sin\frac{\phi_i}{2}. \tag{93}$$

Expressing the above as an infinitesimal transformation and writing

$$u_1u_2u_3 \sim 1 - \frac{1}{i\hbar}\boldsymbol{\phi}\cdot\mathbf{J}, \tag{94}$$

we can read out the generators of these rotations to be

$$\mathbf{J} = \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{95}$$

Thus, spin of a particle is also the generator of rotations in this space.

To demonstrate that spin indeed is the generator of rotations we shall now demonstrate that we can construct the Hamiltonian by rotating it using the implicit calculus in Section II. Let the magnetic field point in the $\hat{\mathbf{z}}$ direction, such that the Hamiltonian has the form

$$H(0) = -H_0\sigma_3 = -H_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{96}$$

A general rotation of the Hamiltonian is governed by the commutation relation

$$\frac{dH}{d\boldsymbol{\theta}} = \frac{1}{i\hbar}\big[H, \mathbf{J}\big] \tag{97}$$

and a rotation about the $\hat{\mathbf{y}}$ axis is given by

$$\frac{dH}{d\theta} = \frac{1}{i\hbar}\big[H, J_y\big]. \tag{98}$$

Using Eq. (95) we have

$$\frac{dH}{d\theta} = \frac{1}{2i}\big[H_0\sigma_3, \sigma_2\big] = H_0\sigma_1 \tag{99}$$

and

$$\frac{d^2H}{d\theta^2} = \frac{1}{2i}\big[H_0\sigma_1, \sigma_2\big] = H_0\sigma_3. \tag{100}$$

The higher derivatives can be evaluated as,

$$\frac{d^nH}{d\theta^n} = \begin{cases} +H_0\sigma_1, & n = 1, 5, \ldots, \\ +H_0\sigma_3, & n = 2, 6, \ldots, \\ -H_0\sigma_1, & n = 3, 7, \ldots, \\ -H_0\sigma_3, & n = 4, 8, \ldots, \end{cases} \tag{101}$$

using Eq. (95) iteratively. Then, starting from the Taylor expansion

$$H(\theta) = H(0) + \frac{\theta}{1!}\left\{\frac{dH}{d\theta}\right\}_0 + \frac{\theta^2}{2!}\left\{\frac{d^2H}{d\theta^2}\right\}_0 + \ldots \tag{102}$$

and Eq. (101) for the derivatives we obtain

$$\begin{aligned} H(\theta) &= -H_0\sigma_3\left(1 - \frac{\theta^2}{2!} + \ldots\right) + H_0\sigma_1\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \ldots\right) \\ &= -H_0\big[\sigma_3\cos\theta - \sigma_1\sin\theta\big], \\ &= -H_0\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \end{aligned} \tag{103a}$$

which is the Hamiltonian in Eq. (75). Thus, it is indeed consistent. This serves as the third example in Section II A illustrating the role of commutation relation as a vriation, in this case generated by rotation about y axis.

V. COMPOSITION OF GENERATORS

A third role of commutation relation is that they provide the algebra for the composition of generators [2]. The structural features of the composition properties discussed in this section is inspired from Ref. [8]. Consider the infinitesimal Poincaré transformation

$$t' = t - \delta t, \tag{104a}$$

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\epsilon} - \delta\boldsymbol{\omega}\times\mathbf{r} - t\delta\mathbf{v}, \tag{104b}$$

where δt is a rigid translation in time, $\delta\boldsymbol{\epsilon}$ is a rigid translation in space, $\delta\boldsymbol{\omega}\times\mathbf{r}$ is a rigid rotation in space, and $t\delta\mathbf{v}$ is an infinitesimal Galilean relative velocity. The unitary operator associated with the above transformation is written in terms of the generators in Table II as

$$U = 1 + \frac{i}{\hbar}G, \tag{105}$$

where

$$G = -H\delta t + \mathbf{P}\cdot\delta\boldsymbol{\epsilon} + \mathbf{J}\cdot\delta\boldsymbol{\omega} + \mathbf{N}\cdot\delta\mathbf{v} + \hbar\delta\phi \tag{106}$$

with an associated infinitesimal action on a state $\langle\ |$ governed by

$$\langle\ |' = \langle\ |\left(1 - \frac{i}{\hbar}G\right). \tag{107}$$

Here $\delta\phi$ takes care of the fact that a unitary operator is unspecified up to a phase.

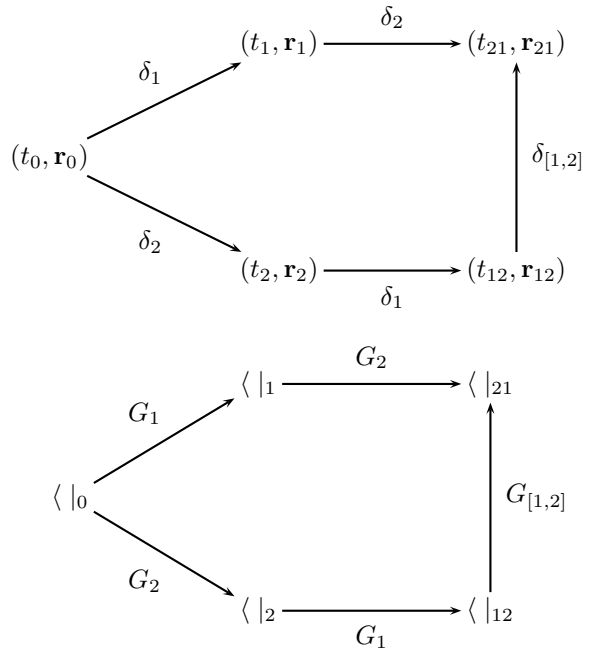

FIG. 1. Composition of successive Poincaré transformations.

Successive infinitesimal transformations illustrated in FIG. 1 can be used to derive

$$
\begin{aligned}
t_{21} &= t_0 - \delta t_1 - \delta t_2, && \text{(108a)}\\
\mathbf{r}_{21} &= \mathbf{r}_0 - \delta_1\boldsymbol{\epsilon} - \delta_2\boldsymbol{\epsilon} - \delta_1\boldsymbol{\omega}\times\mathbf{r}_0 - \delta_2\boldsymbol{\omega}\times\mathbf{r}_0\\
&\quad -t_0\delta_1\mathbf{v} - t_0\delta_2\mathbf{v} - t_0\,\delta_2\boldsymbol{\omega}\times\delta_1\mathbf{v} - \delta_1 t\delta_2\mathbf{v}\\
&\quad -\delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\epsilon} - \delta_2\boldsymbol{\omega}\times(\delta_1\boldsymbol{\omega}\times\mathbf{r}_0), && \text{(108b)}
\end{aligned}
$$

and

$$
\begin{aligned}
t_{12} &= t_0 - \delta t_2 - \delta t_1, && \text{(109a)}\\
\mathbf{r}_{12} &= \mathbf{r}_0 - \delta_2\boldsymbol{\epsilon} - \delta_1\boldsymbol{\epsilon} - \delta_2\boldsymbol{\omega}\times\mathbf{r}_0 - \delta_1\boldsymbol{\omega}\times\mathbf{r}_0\\
&\quad -t_0\delta_2\mathbf{v} - t_0\delta_1\mathbf{v} - t_0\,\delta_1\boldsymbol{\omega}\times\delta_2\mathbf{v} - \delta_2 t\delta_1\mathbf{v}\\
&\quad -\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\epsilon} - \delta_1\boldsymbol{\omega}\times(\delta_2\boldsymbol{\omega}\times\mathbf{r}_0). && \text{(109b)}
\end{aligned}
$$

The composition properties of the Poincaré transformation are derived by subtracting the above transformations,

$$
\begin{aligned}
\delta_{[1,2]}t &= 0, && \text{(110a)}\\
\delta_{[1,2]}\boldsymbol{\varepsilon} &= (\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon} - \delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\varepsilon})\\
&\quad +(\delta_1\mathbf{v}\delta_2 t - \delta_2\mathbf{v}\delta_1 t), && \text{(110b)}\\
\delta_{[1,2]}\boldsymbol{\omega} &= \delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega}, && \text{(110c)}\\
\delta_{[1,2]}\mathbf{v} &= (\delta_1\mathbf{v}\times\delta_2\boldsymbol{\omega} - \delta_2\mathbf{v}\times\delta_1\boldsymbol{\omega}). && \text{(110d)}
\end{aligned}
$$

To derive the commutation relations we use

$$
\frac{1}{i\hbar}\big[G_1, G_2\big] = \delta_{[1,2]}G, \qquad \text{(111)}
$$

which reads

$$
\begin{aligned}
&\frac{1}{i\hbar}\Big[-H\delta_1 t + \delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_1\boldsymbol{\omega}\cdot\mathbf{J} + \delta_1\mathbf{v}\cdot\mathbf{N},\\
&\qquad -H\delta_2 t + \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_2\boldsymbol{\omega}\cdot\mathbf{J} + \delta_2\mathbf{v}\cdot\mathbf{N}\Big]\\
&= -H\delta_{[1,2]}t + \mathbf{P}\cdot\delta_{[1,2]}\boldsymbol{\varepsilon} + \mathbf{J}\cdot\delta_{[1,2]}\boldsymbol{\omega} + \mathbf{N}\cdot\delta_{[1,2]}\mathbf{v}, \qquad \text{(112)}
\end{aligned}
$$

where we used

$$
\delta_{[1,2]}\phi = 0 \qquad \text{(113)}
$$

and refer to discussion in Ref. [2] for further discussion on this. Then, using Eqs. (110) in Eq. (112) we obtain

$$
\begin{aligned}
&\frac{1}{i\hbar}\Big[-H\delta_1 t + \delta_1\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_1\boldsymbol{\omega}\cdot\mathbf{J} + \delta_1\mathbf{v}\cdot\mathbf{N},\\
&\qquad -H\delta_2 t + \delta_2\boldsymbol{\varepsilon}\cdot\mathbf{P} + \delta_2\boldsymbol{\omega}\cdot\mathbf{J} + \delta_2\mathbf{v}\cdot\mathbf{N}\Big]\\
&= -H\,0\\
&\quad +\mathbf{P}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon} - \delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\varepsilon} + \delta_1\mathbf{v}\delta_2 t - \delta_2\mathbf{v}\delta_1 t)\\
&\quad +\mathbf{J}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega})\\
&\quad +\mathbf{N}\cdot(\delta_1\mathbf{v}\times\delta_2\boldsymbol{\omega} - \delta_2\mathbf{v}\times\delta_1\boldsymbol{\omega}). \qquad \text{(114)}
\end{aligned}
$$

Since the commutation relations in Eq. (114) are true for arbitrary infinitesimal transformations we can compare their coefficients to read out the following commutation relations. The composition rules with respect to Hamiltonian are given by

$$
\begin{aligned}
\frac{1}{i\hbar}\big[H\delta_1 t, H\delta_2 t\big] &= 0, && \text{(115a)}\\
\frac{1}{i\hbar}\big[\mathbf{P}\cdot\delta_1\boldsymbol{\varepsilon}, H\delta_2 t\big] &= 0, && \text{(115b)}\\
\frac{1}{i\hbar}\big[\mathbf{J}\cdot\delta_1\boldsymbol{\omega}, H\delta_2 t\big] &= 0, && \text{(115c)}\\
\frac{1}{i\hbar}\big[\mathbf{N}\cdot\delta_1\mathbf{v}, H\delta_2 t\big] &= -\delta_2 t\,\mathbf{P}\cdot\delta_1\mathbf{v}. && \text{(115d)}
\end{aligned}
$$

The composition rules with respect to linear momentum are given by

$$
\begin{aligned}
\frac{1}{i\hbar}\big[H\delta_1 t, \mathbf{P}\cdot\delta_2\boldsymbol{\varepsilon}\big] &= 0, && \text{(116a)}\\
\frac{1}{i\hbar}\big[\mathbf{P}\cdot\delta_1\boldsymbol{\varepsilon}, \mathbf{P}\cdot\delta_2\boldsymbol{\varepsilon}\big] &= 0, && \text{(116b)}\\
\frac{1}{i\hbar}\big[\mathbf{J}\cdot\delta_1\boldsymbol{\omega}, \mathbf{P}\cdot\delta_2\boldsymbol{\varepsilon}\big] &= \mathbf{P}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon}), && \text{(116c)}\\
\frac{1}{i\hbar}\big[\mathbf{N}\cdot\delta_1\mathbf{v}, \mathbf{P}\cdot\delta_2\boldsymbol{\varepsilon}\big] &= -H\,\delta_1\mathbf{v}\cdot\delta_2\boldsymbol{\varepsilon}. && \text{(116d)}
\end{aligned}
$$

The composition rules with respect to angular momentum are given by

$$
\begin{aligned}
\frac{1}{i\hbar}\big[H\delta_1 t, \mathbf{J}\cdot\delta_2\boldsymbol{\omega}\big] &= 0, && \text{(117a)}\\
\frac{1}{i\hbar}\big[\mathbf{P}\cdot\delta_1\boldsymbol{\varepsilon}, \mathbf{J}\cdot\delta_2\boldsymbol{\omega}\big] &= -\mathbf{P}\cdot(\delta_2\boldsymbol{\omega}\times\delta_1\boldsymbol{\varepsilon}), && \text{(117b)}\\
\frac{1}{i\hbar}\big[\mathbf{J}\cdot\delta_1\boldsymbol{\omega}, \mathbf{J}\cdot\delta_2\boldsymbol{\omega}\big] &= \mathbf{J}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\omega}), && \text{(117c)}\\
\frac{1}{i\hbar}\big[\mathbf{N}\cdot\delta_1\mathbf{v}, \mathbf{J}\cdot\delta_2\boldsymbol{\omega}\big] &= -\mathbf{N}\cdot(\delta_2\boldsymbol{\omega}\times\delta_1\mathbf{v}), && \text{(117d)}
\end{aligned}
$$

and the composition rules with respect to boost are given by

$$\frac{1}{i\hbar}\big[H\delta_1 t, \mathbf{N}\cdot\delta_2\mathbf{v}\big] = \delta_1 t\,\mathbf{P}\cdot\delta_2\mathbf{v}, \tag{118a}$$

$$\frac{1}{i\hbar}\big[\mathbf{P}\cdot\delta_1\boldsymbol{\varepsilon}, \mathbf{N}\cdot\delta_2\mathbf{v}\big] = H\,\delta_1\boldsymbol{\varepsilon}\cdot\delta_2\mathbf{v}, \tag{118b}$$

$$\frac{1}{i\hbar}\big[\mathbf{J}\cdot\delta_1\boldsymbol{\omega}, \mathbf{N}\cdot\delta_2\mathbf{v}\big] = \mathbf{N}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\mathbf{v}), \tag{118c}$$

$$\frac{1}{i\hbar}\big[\mathbf{N}\cdot\delta_1\mathbf{v}, \mathbf{N}\cdot\delta_2\mathbf{v}\big] = 0. \tag{118d}$$

The role of commutation relation as a variation can be read out from these relations. This is achieved by factoring out the variation δ_1 and replacing $\delta_2 \to \delta$ in each of the above relation. To avoid dyadic (or tensor) constructions we will prefer to express the variation in the operator X due to an infinitesimal transformation $\delta\lambda$ generated by the generator G in Eq. (3) as

$$\delta X = \frac{1}{i\hbar}\big[X, G\delta\lambda\big]. \tag{119}$$

The variation in the generators with respect to an infinitesimal variation in time are given by

$$\delta H = \frac{1}{i\hbar}\big[H, H\delta t\big] = 0, \tag{120a}$$

$$\delta\mathbf{P} = \frac{1}{i\hbar}\big[\mathbf{P}, H\delta t\big] = 0, \tag{120b}$$

$$\delta\mathbf{J} = \frac{1}{i\hbar}\big[\mathbf{J}, H\delta t\big] = 0, \tag{120c}$$

$$\delta\mathbf{N} = \frac{1}{i\hbar}\big[\mathbf{N}, H\delta t\big] = -\delta t\,\mathbf{P}. \tag{120d}$$

The variation in the generators with respect to an infinitesimal rigid translation in space are given by

$$\delta H = \frac{1}{i\hbar}\big[H, \mathbf{P}\cdot\delta\boldsymbol{\varepsilon}\big] = 0, \tag{121a}$$

$$\delta\mathbf{P} = \frac{1}{i\hbar}\big[\mathbf{P}, \mathbf{P}\cdot\delta\boldsymbol{\varepsilon}\big] = 0, \tag{121b}$$

$$\delta\mathbf{J} = \frac{1}{i\hbar}\big[\mathbf{J}, \mathbf{P}\cdot\delta\boldsymbol{\varepsilon}\big] = \delta\boldsymbol{\varepsilon}\times\mathbf{P}, \tag{121c}$$

$$\delta\mathbf{N} = \frac{1}{i\hbar}\big[\mathbf{N}, \mathbf{P}\cdot\delta\boldsymbol{\varepsilon}\big] = -H\,\delta\boldsymbol{\varepsilon}. \tag{121d}$$

The variation in the generators with respect to an infinitesimal rigid rotation in space are given by

$$\delta H = \frac{1}{i\hbar}\big[H, \mathbf{J}\cdot\delta\boldsymbol{\omega}\big] = 0, \tag{122a}$$

$$\delta\mathbf{P} = \frac{1}{i\hbar}\big[\mathbf{P}, \mathbf{J}\cdot\delta\boldsymbol{\omega}\big] = \delta\boldsymbol{\omega}\times\mathbf{P}, \tag{122b}$$

$$\delta\mathbf{J} = \frac{1}{i\hbar}\big[\mathbf{J}, \mathbf{J}\cdot\delta\boldsymbol{\omega}\big] = \delta\boldsymbol{\omega}\times\mathbf{J}, \tag{122c}$$

$$\delta\mathbf{N} = \frac{1}{i\hbar}\big[\mathbf{N}, \mathbf{J}\cdot\delta\boldsymbol{\omega}\big] = \delta\boldsymbol{\omega}\times\mathbf{N}, \tag{122d}$$

and the variation in the generators with respect to an infinitesimal boost are given by

$$\delta H = \frac{1}{i\hbar}\big[H, \mathbf{N}\cdot\delta\mathbf{v}\big] = \mathbf{P}\cdot\delta\mathbf{v}, \tag{123a}$$

$$\delta\mathbf{P} = \frac{1}{i\hbar}\big[\mathbf{P}, \mathbf{N}\cdot\delta\mathbf{v}\big] = H\,\delta\mathbf{v}, \tag{123b}$$

$$\delta\mathbf{J} = \frac{1}{i\hbar}\big[\mathbf{J}, \mathbf{N}\cdot\delta\mathbf{v}\big] = \delta\mathbf{v}\times\mathbf{N}, \tag{123c}$$

$$\delta\mathbf{N} = \frac{1}{i\hbar}\big[\mathbf{N}, \mathbf{N}\cdot\delta\mathbf{v}\big] = 0. \tag{123d}$$

The role of commutation relations as uncertainty relations associated with Hermitian operators is obtained by factoring out the δ variations in the above relations and then using Robertson's generalization of Heisenberg's uncertainty relations in Eq. (44).

VI. SUCCESSIVE MEASUREMENTS

The quantum essence of the interaction of a spin of a particle in a magnetic field can be invoked in a Stern-Gerlach experiment that consists of a beam of particles passing through a non-uniform magnetic field. Let us choose the y-axis along the direction of the beam. As the beam enters a region of magnetic field oriented perpendicular to the axis of the beam, each particle in the beam experiences a magnetic force, and a magnetic torque. In response to the torque each particle precesses and in response to the force each particle deflects. The distribution of the particles during the deflection of the beam is probabilistic, it is quantized and explained by the principles of quantum mechanics in contrast to what might be expected classically. Further, the projection of the angular momentum of the particles in each of the deflected beam can be predicted based on the initial state of the beam while the projection of the angular momentum perpendicular to the magnetic field are unpredictable and thus random, irrespective of the initial configuration of the beam. If j denotes the spin of the particles in the beam, the beam splits $(2j+1)$-ways, in such a way that the particles in each of the splitted beam has a definite value for the projection of the angular momentum in the direction of $\mathbf{B}$. These quantum mechanical principles also confine the angular momentum j to be integer or half-integer multiples of the rationalized Planck's constant $\hbar$.

The particles in one of the split beam that deflected along the direction of the magnetic field is described by the eigenstate $|+;\theta\rangle$ and the associated projection operator

$$|+;\theta\rangle\langle+;\theta|, \tag{124}$$

where θ is given by $\hat{\mathbf{z}}\cdot\mathbf{B} = B\cos\theta$ and is reminder that this quantity depends on the projection of the angular momentum in the direction of the magnetic field. Similarly, the projection operator

$$|-;\theta\rangle\langle-;\theta|, \tag{125}$$

selects the particles in the eigenstate $|-;\theta\rangle$ that deflected in the opposite direction of the magnetic field. The deflection and the split of the beam allow for the following four selection acts: First involves selecting by projecting $|+;\theta\rangle\langle+;\theta|$ and stopping $|-;\theta\rangle\langle-;\theta|$, Second involves selecting by projecting $|-;\theta\rangle\langle-;\theta|$ and stopping $|+;\theta\rangle\langle+;\theta|$, third involves stopping both the beams, and the fourth involves allowing both the beams. We observe that allowing both the beams involves the sum of the projections that equates to identity,

$$|+;\theta\rangle\langle+;\theta| + |-;\theta\rangle\langle-;\theta| = 1, \tag{126}$$

using completeness of the eigenstates. We also have

$$|+;\theta\rangle = \cos\frac{\theta}{2}|+;0\rangle + \sin\frac{\theta}{2}|-;0\rangle, \tag{127}$$

$$|-;\theta\rangle = -\sin\frac{\theta}{2}|+;0\rangle + \cos\frac{\theta}{2}|-;0\rangle. \tag{128}$$

Successive Stern-Gerlach setups in sequence will involve projecting the angular momentum along a magnetic field oriented at angle θ_2, after it has been projected along the magnetic field oriented at angle θ_1. For simplicity, and at the loss of considerable generality, let us confine to selecting by projecting $|+;\theta_i\rangle\langle+;\theta_i|$ and stopping $|-;\theta_i\rangle\langle-;\theta_i|$, at each of the i-th stage of measurement in sequence. This sequential projection of angular momentum along magnetic fields oriented along θ_1 and θ_2 respectively, is summarized as

$$\boxed{\begin{array}{c}\text{measure } J_z(\theta_1)\\ \text{select } |+;\theta_1\rangle\langle+;\theta_1|\end{array}} \rightarrow \boxed{\begin{array}{c}\text{measure } J_z(\theta_2)\\ \text{select } |+;\theta_2\rangle\langle+;\theta_2|\end{array}} \tag{129}$$

where the arrow in between stands for a dynamical evolution of angular momentum generated by the operator J_y that is associated to a rotation about the y-axis. In the sequential projection of angular momentum along two respective magnetic fields, as in Eq. (129), the probability of finding a particle in the eigenstate $|+;\theta_2\rangle$ after the knowledge that it was selected to be in the eigenstate $|+;\theta_1\rangle$ is given by

$$p_2 = |\langle+;\theta_1|+;\theta_2\rangle|^2. \tag{130}$$

Using

$$\begin{aligned}\langle+;\theta_1|+;\theta_2\rangle &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ &= \cos\left(\frac{\theta_1-\theta_2}{2}\right)\end{aligned} \tag{131}$$

we have

$$p_2 = \cos^2\left(\frac{\theta_1-\theta_2}{2}\right). \tag{132}$$

A. Measurement disturbs the system

Let us have an initial measurement that selects the + beam, with the magnetic field directed along the $\hat{\mathbf{z}}$ direction, that is, $\theta_1 = 0$. We next perform another measurement that selects the + beam, with the magnetic field directed along the $-\hat{\mathbf{z}}$ direction, that is, $\theta_2 = \pi$. The notation we shall use for the probability in this two successive measurements is

$$\begin{aligned}&p_{N=1}(|+;0\rangle\langle+;0|\\ &\qquad\downarrow\\ &\qquad|+;\pi\rangle\langle+;\pi|) = 0.\end{aligned} \tag{133}$$

We denote this as $N = 1$ because the initial measurement is interpreted as preparing the system.

The two measurements above can be interpreted as the 'source' and 'detector'. Let us consider a configuration involving three measurements, each selecting the + beam in succession, and the magnetic field directions being $\theta_1 = 0$, $\theta_2 = \pi/2$, and $\theta_3 = \pi$.

$\underline{N = 2\text{: selecting + beams}}$

$$\begin{aligned}&p_{N=2}(|+;0\rangle\langle+;0|\\ &\qquad\downarrow\\ &\qquad|+;\tfrac{\pi}{2}\rangle\langle+;\tfrac{\pi}{2}| \qquad \text{(select + beam)}\\ &\qquad\downarrow\\ &\qquad|+;\pi\rangle\langle+;\pi|) = \left(\cos\frac{\pi}{4}\right)^4.\end{aligned} \tag{134}$$

This configuration should be contrasted with the closely related setup in which the beam passes through the same three magnetic fields, with the difference that now we do not block any of the beams, that is, we allow all the beams to pass through, or we select all beams. Since we are allowing all the beams, equivalent to saying that we are allowing all the projections of a state, the system is able to reconstruct the complete original state. At the heart of this reconstruction is the completeness relation of the eigenstates in Eq. (126).

$\underline{N = 2\text{: selecting ALL beams}}$

$$\begin{aligned}&p_{N=2}(|+;0\rangle\langle+;0|\\ &\qquad\downarrow\\ &|+;\tfrac{\pi}{2}\rangle\langle+;\tfrac{\pi}{2}| \;+\; |-;\tfrac{\pi}{2}\rangle\langle-;\tfrac{\pi}{2}| \quad \text{(select both beams)}\\ &\qquad\downarrow\\ &\qquad|+;\pi\rangle\langle+;\pi|) = 0.\end{aligned} \tag{135}$$

The observation in the case $N = 2$ is the content of the statement: measurement (with partial selection of states) disturbs the system. It brings out the difference between a partial selection (+ beam is selected and − beam is blocked) versus selecting all states (both + beam and − beam is selected). In particular, selecting all the states does not disturb the system.

B. Quantum Zeno effect

We can imagine $(N+1)$ measurements that selects the $+$ beam successively, with magnetic field direction shifted in each successive measurement by an equal amount such that the direction of the magnetic field in each intermediate position is

$$0, \frac{\pi}{N}, \frac{2\pi}{N}, \ldots, \pi. \tag{136}$$

During the measurement we contrast between making a selection of the $+$ beam versus selecting all states.

$$\begin{aligned} & p_{N=3}(|+;0\rangle\langle+;0| \\ & \quad\downarrow \\ & |+;\tfrac{\pi}{3}\rangle\langle+;\tfrac{\pi}{3}| \qquad \text{(select + beam)} \\ & \quad\downarrow \\ & |+;\tfrac{2\pi}{3}\rangle\langle+;\tfrac{2\pi}{3}| \qquad \text{(select + beam)} \\ & \quad\downarrow \\ & |+;\pi\rangle\langle+;\pi|) = \left(\cos\frac{\pi}{6}\right)^6. \end{aligned} \tag{137}$$

In contrast if we select all the beams each time the particles pass through a magnetic field we obtain a null result, as if the magnetic field did not disturb the states. For $N=3$,

$$\begin{aligned} & p_{N=3}(|+;0\rangle\langle+;0| \\ & \quad\downarrow \\ & |+;\tfrac{\pi}{3}\rangle\langle+;\tfrac{\pi}{3}| \;+\; |-;\tfrac{\pi}{3}\rangle\langle-;\tfrac{\pi}{3}| \quad \text{(select both beams)} \\ & \quad\downarrow \\ & |+;\tfrac{2\pi}{3}\rangle\langle+;\tfrac{2\pi}{3}| \;+\; |-;\tfrac{2\pi}{3}\rangle\langle-;\tfrac{2\pi}{3}| \quad \text{(select both beams)} \\ & \quad\downarrow \\ & |+;\pi\rangle\langle+;\pi|) = 0. \end{aligned} \tag{138}$$

This can be easily extended for N successive measurements.

$$p_N(+\text{ selections}) = \left(\cos\frac{\pi}{2N}\right)^{2N}. \tag{139}$$

Again, selecting all the states during the measurements does not disturb the system.

$$p_N(\text{ALL selections}) = 0. \tag{140}$$

In the extreme limit of $N\to\infty$ we have

$$\lim_{N\to\infty}\left(\cos\frac{\pi}{2N}\right)^{2N} = 1 \tag{141}$$

we have

$$p_\infty(+\text{ selections}) = 1 \tag{142}$$

and

$$p_\infty(\text{ALL selections}) = 0. \tag{143}$$

Thus, measurement with partial selection repeated continuously, that is infinite times, does not disturb the system, even though each of the single measurement with partial selection disturbs the system. In the context of quantum computers we are interested in the robustness of a system continuing in an eigenstate in the presence of interactions with surroundings. If one assumes rotation to be such an interaction, the suggestion is that repeating the measurement infinite times continuously (with selection of the desired eigenstate during each measurement) will lead to error correction.

VII. SUMMARY

We emphasized and illustrated three distinct roles played by commutation relation. To summarize the content of this article we point them out in the context of a rotation in space (generated by angular momentum) followed by a translation in space (generated by linear momentum). The composition rules or the algebra for such sequential operations are contained in the commutation relations

$$\frac{1}{i\hbar}\big[\mathbf{J}\cdot\delta_1\boldsymbol{\omega}, \mathbf{P}\cdot\delta_2\boldsymbol{\varepsilon}\big] = \mathbf{P}\cdot(\delta_1\boldsymbol{\omega}\times\delta_2\boldsymbol{\varepsilon}). \tag{144}$$

The above commutation relation is associated with a variation in the angular momentum $\mathbf{J}$ operator due to a translation in space as

$$\delta\mathbf{J} = \delta\boldsymbol{\varepsilon}\times\mathbf{P} \tag{145}$$

and with a variation in the linear momentum $\mathbf{P}$ operator due to a rotation in space as

$$\delta\mathbf{P} = \delta\boldsymbol{\omega}\times\mathbf{P}. \tag{146}$$

Observe how $\mathbf{P}$ and $\mathbf{J}$ change their roles as operators and generators. A measurement disturbs the system. That is, the angular momentum operator is disturbed due to a measurement involving translation in space, as per Eq. (145). Similarly, the linear momentum operator is disturbed due to a measurement involving rotation in space, as per Eq. (146). The commutation relations in Eq. (144) are expressed in index notation as

$$\frac{1}{i\hbar}\big[J_i, P_j\big] = \varepsilon_{ijk}P_k. \tag{147}$$

Using the uncertainty relations for two Hermitian operators derived in Eq. (44) we obtain

$$\frac{\hbar}{2}\varepsilon_{ijk}|\langle P_k\rangle| \le (\delta J_i)(\delta P_j), \tag{148}$$

in terms of the mean square deviations in operators defined in Eqs. (29). Multiplying by ε_{ijm} on both sides of this equation allows us to isolate the momentum operator in the above relation as

$$\hbar|\langle P_k\rangle| \le \varepsilon_{kij}(\delta J_i)(\delta P_j). \tag{149}$$

A variation in an operator due to a generator amounts to a lower bound on the product of deviations from their average values. That is, uncertainty relations are a direct consequence of the role of measurements disturbing a system.

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